

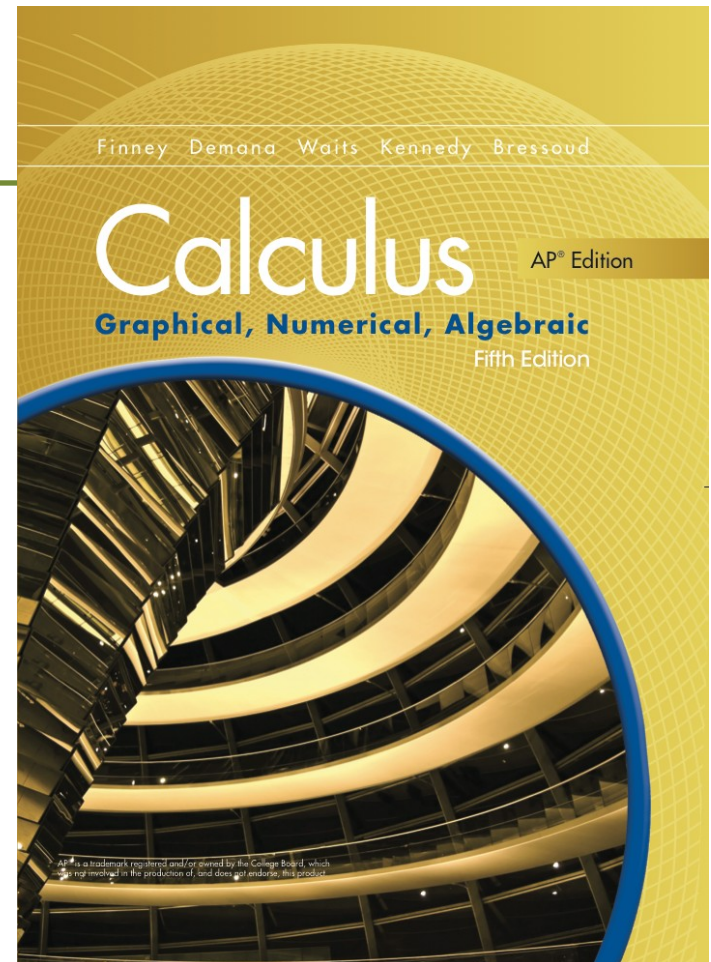
Chapter 10

Infinite Series



Section 10.1

Power Series



Quick Review

Find the first four terms and the 30th term of the sequence

$$\{ u_n \}_{n=1}^{\infty} = \{ u_1, u_2, \dots, u_n, \dots \} .$$

$$1. u_n = \frac{4}{n+2}$$

$$2. u_n = \frac{(-1)^n}{n}$$

The following sequences are geometric. Find **(a)** the common ratio, **(b)** the tenth term, and **(c)** a rule for the n th term.

$$3. \{ 2, 6, 18, 54, \dots \}$$

$$4. \{ 8, -4, 2, -1, \dots \}$$

Quick Review

Determine $\lim_{n \rightarrow \infty} a_n$.

$$5. a_n = \frac{1 - n}{n^2}$$

$$6. a_n = \left(1 + \frac{1}{n}\right)^n$$

$$7. a_n = \frac{\ln(n+1)}{n}$$

$$8. a_n = 2 - \frac{1}{n}$$

Quick Review Solutions

Find the first four terms and the 30th term of the sequence

$$\{u_n\}_{n=1}^{\infty} = \{u_1, u_2, \dots, u_n, \dots\}.$$

$$1. u_n = \frac{4}{n+2} \quad 4/3, 1, 4/5, 2/3, 1/8$$

$$2. u_n = \frac{(-1)^n}{n} \quad -1, 1/2, -1/3, 1/4, 1/30$$

The following sequences are geometric. Find **(a)** the common ratio, **(b)** the tenth term, and **(c)** a rule for the n th term.

$$3. \{2, 6, 18, 54, \dots\} \quad (a) 3 \quad (b) 39,366 \quad (c) a_n = 2(3)^{n-1}$$

$$4. \{8, -4, 2, -1, \dots\} \quad (a) -1/2 \quad (b) -1/64 \quad (c) a_n = 8(-1/2)^{n-1}$$

Quick Review Solutions

Determine $\lim_{n \rightarrow \infty} a_n$.

$$5. a_n = \frac{1-n}{n^2} \quad 0$$

$$6. a_n = \left(1 + \frac{1}{n}\right)^n \quad e$$

$$7. a_n = \frac{\ln(n+1)}{n} \quad 0$$

$$8. a_n = 2 - \frac{1}{n} \quad 2$$

What you'll learn about

- Geometric series
- Convergence and divergence of series; the sequence of partial sums
- Power series
- Representing functions by power series
- Differentiating and integrating power series term by term
- Power series for

... and why $\frac{1}{1-x}, \frac{1}{1+x}, \ln(1+x), \frac{1}{1+x^2}, \tan^{-1} x, e^x$

Power series are important in understanding the physical universe and can be used to represent functions.

Infinite Series

An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots, \quad \text{or} \quad \sum_{k=1}^{\infty} a_k.$$

The numbers $a_1, a_2, \dots$ are the **terms** of the series;

a_n is the ***n*th term**

Example Identifying a Divergent Series

Does the series $2 - 2 + 2 - 2 + 2 - 2 + \dots$ converge?

If this infinite series has a sum it has to be the limit of its partial sums,
 $2, 0, 2, 0, 2, \dots$

Since this sequence has no limit, the series has no sum. It diverges.

Example Identifying a Convergent Series

Does the series $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \dots + \frac{7}{10^n} + \dots$ converge?

The sequence of partial sums, written in decimal form is
0.7, 0.77, 0.777, ...

This sequence has a limit $0.\overline{7}$ or $7/9$. This series converges
to the sum $7/9$.

Infinite Series

If the **infinite series** $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \dots + a_k + \dots$

converges, then $\lim_{k \rightarrow \infty} a_k = 0$.

This means that if $\lim_{k \rightarrow \infty} a_k \neq 0$ the series must diverge.

Geometric Series

The geometric series $a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$

converges to the sum $\frac{a}{(1-r)}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

Example Analyzing Geometric Series

Tell whether each series converges or diverges. If it converges, give its sum.

$$\sum_{n=1}^{\infty} 5 \left(\frac{1}{2} \right)^{n-1}$$

The first term is $a = 5$ and $r = 1/2$. The series converges to

$$\frac{5}{1 - (1/2)} = 10$$

Example Analyzing Geometric Series

Tell whether each series converges or diverges. If it converges, give its sum.

$$\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$$

First term is $a = (3/5)^0 = 1$ and $r = 3/5$. The series converges to

$$\frac{1}{1 - (3/5)} = \frac{5}{2}.$$

Power Series

An expression of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$

is a **power series centered at $x = 0$** . An expression of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \dots + c_n (x - a)^n + \dots$$

is a **power series centered at $x = a$** . The term $c_n (x - a)^n$ is the **n th term**; the number a is the **center**.

Example Finding a Power Series by Differentiation

Given that $1/(1-x)$ is represented by the power series $1 + x + x^2 + x^3 + \dots + x^n + \dots$ on the interval $(-1, 1)$, find a power series to represent $1/(1-x)^2$.

Notice that $1/(1-x)^2$ is the derivative of $1/(1-x)$. To find the power series, differentiate both sides of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + \dots + x^n + \dots)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots$$

Term-by-Term Differentiation

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots + c_n (x-a)^n + \dots$

converges for $|x-a| < R$, then the series

$$\sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \dots + n c_n (x-a)^{n-1} + \dots,$$

obtained by differentiation the series for f term by term, converges for $|x-a| < R$ and represents $f'(x)$ on that interval. If the series for f converges for all x , then so does the series for f' .

Example Finding a Power Series by Integration

Given that $\frac{1}{x+1} = 1 - x + x^2 - x^3 + \dots + (-x)^n + \dots$, $-1 < x < 1$,

find a power series to represent $\ln(1+x)$.

Recall that $1/(1+x)$ is the derivative of $\ln(1+x)$. Integrate the series $1/(1+x)$ to find the series for $\ln(1+x)$.

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \dots + (-x)^n + \dots$$

$$= 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$$

$$\int_0^x \frac{1}{t+1} dt = \int_0^x (1 - t + t^2 - t^3 + \dots + (-1)^n t^n + \dots) dt$$

$$\ln(t+1) \Big|_0^x = \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots + (-1)^n \frac{t^{n+1}}{n+1} + \dots \right]_0^x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots$$

Term-by-Term Integration

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots + c_n (x-a)^n + \dots$

converges for $|x-a| < R$, then the series

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = c_0 (x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots + c_n \frac{(x-a)^{n+1}}{n+1} + \dots,$$

obtained by integrating the series for f term by term, converges for

$|x-a| < R$ and represents $\int_a^x f(t) dt$ on that interval. If the series for f

converges for all x , then so does the series for the integral.